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Supplement to the Paper, "Orders in QF–3 Rings"

Vol. **14**, No. 1 (1970), in the article, "Orders in QF–3 Rings," by Charles Vinsonhaler, pp. 83-90:

In a recent communication, E.P. Armendariz questioned a step in the proof of Lemma 7 in my paper "Orders in QF–3 rings" [1]. The assertion was the following: Let R be a left Noetherian ring with identity and zero singular ideal. Let Re be a faithful injective left ideal in R, $e^2 = e$, and Re_0 an indecomposable direct summand of Re, $e_0^2 = e_0$. If $M = \oplus \Sigma E(S_\alpha)$ is the sum of the injectives hulls of a representative set of nonisomorphic simple R-modules, then Re_0 is a direct summand of πM, a product of copies of M. Hence, there is a nonzero R-map $f : M \to Re_0$.

Efforts to justify the last sentence led to the following theorem:

THEOREM A. *If R is a left Noetherian ring with identity and $E({}_RR)$, the injective hull of ${}_RR$, is finitely generated, then R is left Artinian.*

Proof of Theorem A. Essentially the proof of Theorem 8 in [1] can be applied. Since R is left Noetherian, we can write R as a direct sum of indecomposable, idempotent generated left ideals. Let Rf be one of these, $f^2 = f$. Let $\phi : Rf \to Rf$ be an R-map, and assume ϕ is not nilpotent. The increasing sequence $\operatorname{Ker} \phi \subsetneq \operatorname{Ker} \phi^2 \subsetneq \cdots$ must terminate, so assume

$$\operatorname{Ker} \phi^n = \operatorname{Ker} \phi^{n+1} = \cdots,$$

for some positive integer n. The restriction $\tilde{\phi}$, of ϕ to $\operatorname{Im} \phi^n$, is then $1 - 1$. We show $\tilde{\phi}$ is onto as follows. Let $I = \operatorname{Im} \phi^n$. By injectivity, $\tilde{\phi} : I \to I$ can be extended to $\beta : E(I) \to E(I)$. The map β will be $1 - 1$, and therefore must be onto. If not, then β would split, producing an infinite ascending chain of direct summands of $E(I) \subseteq E(R)$, contradicting the assumption that $E(R)$ is finitely generated. This assumption can then be applied to the chain $\beta^{-1}(I) \subseteq \beta^{-2}(I) \subseteq \cdots$, to obtain $\beta^{-r+1}(I) = \beta^{-r}(I)$ for some positive integer r. This implies $\beta(I) = I = \phi(I)$. Now ϕ^n also induces an isomorphism of I onto I, with an inverse we will call h. Then $h\phi^n : Rf \to Rf$ is idempotent and, therefore, splits. Since Rf is indecomposable, $h\phi^n$ and hence ϕ, must be $1 - 1$. By an argument similar to that above, showing $\tilde{\phi}$ onto, ϕ is onto. Thus, if J is the Jacobson radical of R, fJf is the Jacobson radical of $fRf = \operatorname{Hom}_R(Rf, Rf)$, and fRf/fJf is a division ring. This implies Rf/Jf is simple by [2, Prop. 1, p. 65]. Furthermore, fRf inherits ascending chain condition from R, and since fJf is nil (nonunits in fRf are nilpotent), fJf is nilpotent by a theorem of Levitski (see [2, p. 199]). Thus Jf is a nilpotent left ideal in R. Let N be the unique maximal nilpotent ideal in R. Then $Jf \subseteq N$ implies $fJf \subseteq fNf$, so that $fJf = fNf$. Therefore, $fRf/fNf = fRf/fJf$ is a division ring, Rf/Nf is a simple R/N (hence R) module, and R/N is semi-simple. Since N is nilpotent, we can obtain a composition series for R, which is then left Artinian.

This theorem generalizes a result of Faith and Walker [5], who showed that a left Noetherian ring, with $E(R/N)$ finitely generated, is left Artinian. In regard to the proof of Lemma 7 in [1], we obtain the following:

COROLLARY. *If R is left Noetherian ring with identity which has a faithful projective injective left ideal Re, $e^2 = e$, then R is left Artinian.*

Proof. Tachikawa [4] has shown that R can be embedded in a product $(Re)^n$ of a finite number of copies of Re. The injective hull $E(R)$ is then contained in this product, and hence is finitely generated.

The above Corollary justifies the proof of Lemma 7 in [1], as the ring R in question will be left Artinian. Note that the hypothesis of zero singular ideal is unnecessary. The Corollary also enables considerable simplification of the characterization of orders in QF–3 rings found in [1]. This is given in

THEOREM B. *R is a left order in a left Artinian QF–3 ring Q if and only if*

(1) *$\sum_{\alpha\in A} E(M_\alpha)$ is injective whenever $\{M_\alpha\}_{\alpha\in A}$ is a set of R-modules with $T(M_\alpha) = 0$.*

(2) *There is a faithful injective R-module $X \subseteq U(R)$ with $T(X) = 0$.*

The notation used is the same as that used in [1]. For any R-module M, $T(M) = \{m \in M \mid bm = 0$ for some b regular in $R\}$. The module $U(R)$ is the unique maximal submodule of $E(R)$ satisfying $T(U(R)/R) = U(R)/R$.

Proof of Theorem B. By Theorem 6 of [1], any order R in a left Artinian QF–3 ring must satisfy conditions (1) and (2). Conversely, if R satisfies (1) and (2), we can use the proof of Theorem 6 to show that $Q = U(R)$ is a left Noetherian quotient ring for R. The module X will be a faithful projective injective left ideal for Q, and the corollary to Theorem A can be applied to show Q is a left Artinian. This in turn implies Q is left QF–3.

Remark. E.P. Armendariz has also pointed out that if my proof of Lemma 7 in [1] were valid, the Lemma would be true in the more general case of R having finite (Goldie) dimension. The corollary to Theorem A above, of course, sheds no light on this question.

References

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